Landau Collision Integral

From Boltzmann $\rightarrow$ Landau:

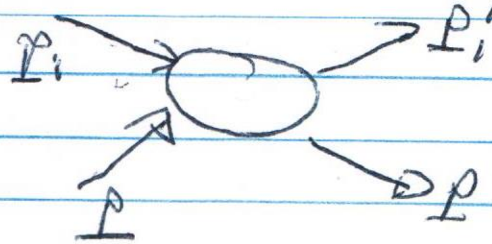
$$\frac{df}{dt} = C(f)$$

$$= \int d\underline{p}_1 \int d\underline{p}' \int d\underline{p}_1' \, W \left(\overset{①}{f(\underline{p}')f(\underline{p}_1')} - \overset{②}{f(\underline{p})f(\underline{p}_1)} \right)$$

$$\underline{p} + \underline{p}_1 = \underline{p}' + \underline{p}_1'$$

and

$$C(f) = -\frac{\partial}{\partial \underline{p}} \cdot \underline{J}(\underline{p})$$



Key point for Coulomb scattering:

$\boxed{\text{scattering is event of } \underline{\text{small}} \text{ momentum transfer}}$ X

i.e. can proceed by small momentum transfer limit of Boltzmann Eqn.

i.e. $\quad \underline{p} \rightarrow \underline{p} + \underline{q}/2$

$\quad \underline{p}' \rightarrow \underline{p}' - \underline{q}/2$

Consider terms in B.E.:

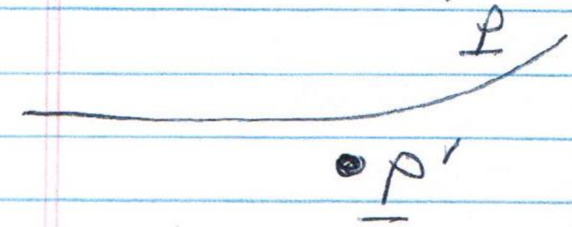
i.e. term ②, with $\underline{p}_1 \to \underline{p}'$ (re-label)

$\Rightarrow$

\# collisions per time between particle with momentum $\underline{p}$ (test) and particle with momentum $\underline{p}'$.

$$\# = \dot{w}\left(\underline{p}+\underline{q}/2,\ \underline{p}'-\underline{q}/2,\ \underline{q}\right) \langle f(\underline{p})\, f(\underline{p}')\rangle\, d^3p'\, d^3q$$

i.e.



$\underline{p} \to$ test particle, scatter-ee

$\underline{p}' \to$ background or field particle, scatter-er.

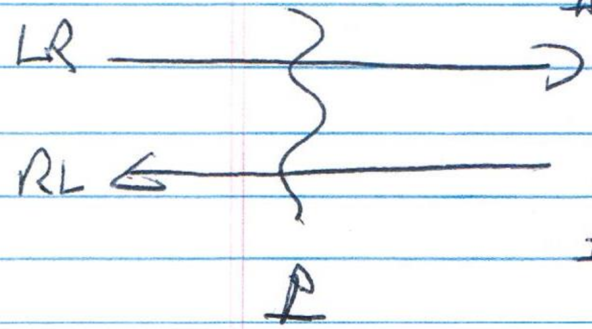
$\underline{q} \equiv$ momentum transfer.

$$C(f) = -\frac{\partial}{\partial \underline{p}} \cdot \underline{J}(\underline{p})$$

easier to construct $\underline{J}(\underline{p})$, scattering current

so

$\underline{J}(\underline{p})$:



Left → Rt
i.e. upshift in momentum

Rt → left
i.e. decrease in momentum

$J(\underline{p}) \rightarrow LR + RL$

mom transfer bckgnd

$$\#LR = \sum_{spec}^{ee,\,ei} \int_{|q|\geq 0} d^3\underline{q} \int d^3p' \int_{\underline{p}-\underline{q}}^{\underline{p}} \Big[W\left(\underline{q};\, \underline{p}+\frac{\underline{q}}{2},\, p'-\frac{\underline{q}}{2}\right) \times \langle f(\underline{p})\rangle \langle f(\underline{p}')\rangle \Big]$$

$$\#RL = \sum_{spec}^{ee,\,ei} \int_{|q|>0} d^3q \int d^3p' \Big[W\left(-\underline{q},\, \underline{p}+\frac{\underline{q}}{2},\, \underline{p}'-\frac{\underline{q}}{2}\right) \times \langle f\left(\underline{p}+\frac{\underline{q}}{2}\right)\rangle \langle f\left(p'-\frac{q}{2}\right)\rangle$$

Now

– detailed balance: $w(\underline{q}) = w(-\underline{q})$

– small deflection:

$$w(\underline{p}+\underline{q}/2, \underline{p}'-\underline{q}/2; \underline{q}) \cong w(\underline{p}, \underline{p}'; \underline{q})$$

so, for small $\underline{q}$:

$$\underline{J}(\underline{p}) = \sum_{\text{spec.}} \int_{|q|>0} d^3\underline{q} \int d^3p' \int_{\underline{p}-\underline{q}}^{\underline{p}} w \Big[\langle f(\underline{p})\rangle \langle f(\underline{p}')\rangle - \langle f(\underline{p}+\underline{q})\rangle \langle f(\underline{p}'-\underline{q})\rangle \Big]$$

expand

$$= [\ \ ] \Big\{ \cancel{\langle f(\underline{p})\rangle \langle f(\underline{p}')\rangle} - \cancel{\langle f(\underline{p})\rangle \langle f(\underline{p}')\rangle} + \underline{q} \cdot \frac{\partial \langle f(\underline{p}')\rangle}{\partial \underline{p}'} \langle f(\underline{p})\rangle - \underline{q} \cdot \frac{\partial \langle f\rangle}{\partial \underline{p}} \langle f(\underline{p}')\rangle \Big\}$$

N.B.: Unsurprisingly, $\langle f\rangle$ gradients set current;

and,

$$[\ \ ] = (\ )\int_{p_\alpha - q_\alpha}^{p_\alpha} \equiv (\ )\, q_\alpha$$

so

$$\boxed{J(p) = \sum_s \int d^3p' \left[\langle f(p)\rangle \frac{\partial}{\partial p'_\beta} \langle f(p')\rangle - \langle f(p')\rangle \frac{\partial \langle f(p)\rangle}{\partial p_\beta} \right] B_{\alpha\beta}}$$

$$B_{\alpha,\beta} = \int d^3q\, \frac{W(q)}{2}\, q_\alpha q_\beta \rightarrow$$ scattering vector $\sim \langle q^2 \rangle$

↳ prevents dbl counting

But

$$W d^3q = |v - v'|\, d\sigma$$

$$= m|q|\, d\sigma$$

so

$$B_{\alpha\beta} = \int \frac{d^3q}{2}\, |v - v'|\, q_\alpha q_\beta\, d\sigma.$$

so

$$\frac{df}{dt} = -\frac{\partial}{\partial \underline{p}} \cdot \underline{J}(\underline{p})$$

$$J(\underline{p}) = \left[-\frac{\partial}{\partial \underline{p}} \cdot \underline{\underline{D}} \langle f(\underline{p}) \rangle + \underline{F} \langle f(\underline{p}) \rangle \right]$$

$\partial / \partial \underline{p}$ pulls thru

$$\underline{\underline{D}} = \sum_s \int d^3p' \langle f(\underline{p}') \rangle \underline{\underline{B}}_{\alpha,\beta}$$

→ diffusion tensor (in velocity)

→ scattering by "field" particles ($\langle f(\underline{p}') \rangle$)
  i.e. test particle diffused by background particles

$$\underline{F} = \sum_s \int d^3p' \frac{\partial \langle f(\underline{p}') \rangle}{\partial p'_\beta} B_{\alpha,\beta}$$

→ drag, dynamical friction

→ friction exerted on test particle by field particles.

N.B.

$$\frac{df}{dt} = -\frac{\partial}{\partial \underline{p}} \cdot \underline{\Gamma}(\underline{p})$$

then for "slowing down":

$$\frac{d}{dt}\langle \underline{p} \rangle = \frac{d}{dt}\int d\underline{p}\, \underline{p}\, f$$

$$= \int d\underline{p} \left[\underline{F} \langle f(\underline{p}) \rangle - \cancel{\frac{\partial}{\partial \underline{p}} \cdot \underline{\underline{D}} \langle f(\underline{p}) \rangle} \right]$$

$$= \int d\underline{p} \left[\underline{F} \langle f(\underline{p}) \rangle \right]$$

controls slowing.

and for "energization":

$$\frac{d}{dt}\left\langle \frac{p^2}{2} \right\rangle = \int d\underline{p} \left[D \langle f(\underline{p}) \rangle - \underline{p} \cdot \underline{F} \langle f(\underline{p}) \rangle \right]$$

increases mean square p.

In Fokker-Planck Formulation:

$$\underline{\underline{D}} = \langle \Delta \underline{p} \, \Delta \underline{p} \rangle / 2\Delta t$$

$$\underline{F} = \left\langle \frac{\Delta \underline{p}}{\Delta t} \right\rangle$$

To simplify:

- $|\underline{q}|$ small, small angle scattering

$\underline{q} \perp \underline{v} - \underline{v}'$ → deflection

so

- $(v_\beta - v'_\beta) B_{\alpha,\beta} = 0$

i.e. $\underline{\underline{B}}$ transverse to $\underline{v} - \underline{v}'$ and depends on $\underline{v} - \underline{v}'$;

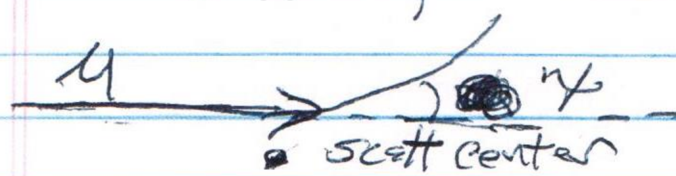
$$B_{\alpha,\beta} = B \left[\delta_{\alpha,\beta} - \frac{(v_\alpha - v'_\alpha)(v_\beta - v'_\beta)}{(v - v')^2} \right]$$

$$B = \frac{1}{2} \int \underline{q}^{\prime 2} |v - v'| \, d\sigma = B_{\alpha,\alpha}$$

For $B_{\alpha,\beta}$:

χ = angle deviation $\underline{v}-\underline{v}'$ (upon collision)

recall



$|q| = \mu\,|v-v'|\,\chi$ ($\sin\chi \sim \chi$)

reduced mass

$\Rightarrow$

$$B = \frac{1}{2}\int d\sigma\, q^2\,|v-v'|$$

$$= \frac{1}{2}\int d\sigma\,(\mu^2|v-v'|^2\chi^2)\,|v-v'|$$

$$= \frac{\mu^2}{2}|v-v'|^3\int \chi^2\, d\sigma$$

but $\sigma_T = \int (1-\cos\chi)\, d\sigma$ → transverse transport cross section

transverse scattering

$$\cong \int d\sigma\, \frac{\chi^2}{2}$$

Now, for Coulomb scattering:

$$d\sigma = 4(ee')^2 d\chi / \mu^2 v_{rel}^4 \chi^3$$

so

$$B = \frac{\cancel{\mu^2}}{2} |v-v'|^2 \int \frac{d\chi}{2} \frac{\cancel{4}(ee')^2 \cancel{\chi^2}}{\cancel{\mu^2} \cancel{v_{rel}^4} \cancel{\chi^3}} \cancel{v_{rel}}$$

$$= \frac{1}{|v_{rel}|} (ee')^2 \int \frac{d\chi}{\chi}$$

$$= \frac{(ee')^2}{|v_{rel}|} \int \frac{d\chi}{\chi}$$

Finally,

$$\begin{cases} B_{\alpha,\beta} = B\left[\delta_{\alpha,\beta} - \dfrac{(v_\alpha - v'_\alpha)(v_\beta - v'_\beta)}{(v-v')^2} \right] \\ B = \dfrac{(ee')^2}{|v_{rel}|} \ln \Lambda \end{cases}$$

→ Coulomb logarithm.

and

$$\begin{cases} C(f)_{\text{Landau}} = -\dfrac{\partial}{\partial \underline{p}} \cdot \underline{J}(p) \\ \\ \underline{J}(p) = \left[-\dfrac{\partial}{\partial \underline{p}} \cdot \underline{\underline{D}} \langle f(p) \rangle + \underline{F} \langle f(p) \rangle \right] \end{cases}$$

$$\begin{cases} \underline{\underline{D}} = \sum_{s} \int d^3p \, \langle f(p') \rangle \underline{\underline{B}}_{\alpha,\beta} \\ \\ \underline{F} = \sum_{s} \int d^3p \, \dfrac{\partial}{\partial p'_{\beta}} \langle f(p') \rangle B_{\alpha,\beta} \end{cases}$$

completes Landau Collision Operator